Faster Convex Hull Computation By Reducing Computatinal Overhead and Time

Ngangbam Herojit Singh , Laiphrakpam Dolendro Singh

Abstract — Finding the convex hull of a point set has applications in research fields as well as industrial tools. This paper presents a pre-processing algorithm for computing convex hull vertices in a 2D spatial point set. Based on the position of extreme points we divide the exterior points into four groups bounded by rectangles (p-Rect). Then inside each p-Rect we recursively find and check the extreme points to verify if they are eligible to be convex hull points or not. The process gives a small set of candidate points for convex hull computation. Efficiency of the algorithm is evaluated with respect to time and space. Performance comparison with other classical algorithms shows that implementation of this pre-processing algorithm significantly improves their performance by reducing computational overhead and time.

Index Terms — Quick Convex Hull, Recursion, *p*-Rect, Point rotation, Preprocessing Algorithm.

1 INTODUCTION

The convex hull (CH) of a set Q of points is the smallest convex polygon P for which each point in Q is either on the boundary of P or in its interior [1]. Its application area includes computer visualization, ray tracing, path finding, visual pattern matching, verification methods and geometry. It is also used as a tool to construct geometric shapes, the GIS applications such as area cut, subdivision of Triangulated Irregular Network (TIN) and DTM generation and area dynamic calculation [2].

Convex hull algorithms are broadly divided into two categories: 1) graph traversal and 2) incremental [3]. The graph traversal algorithms construct CHs by identifying some initial vertices of CH and later finding the remaining points and edges by traversing it in some order. The Graham scan [4], Jarvis march [5] and Monotone chain [6] are such algorithms. Incremental algorithms first find an initial CH and then insert or merge the remaining points, edges or even sub CHs as they are discovered, into current CH sequentially or recursively to obtain the final CH. Quickhull [7] and Divide-and-Conquer [8] belong to this class.

Most of these algorithms scan and process all the points one by one resulting in much computational overhead. Normally, the final convex hull contains only a few points, and most of the points are in the interior of convex hull. Based on this concept we devised a fast preprocessing algorithm which filters out most of the non-convex hull points. The final CH points can be easily found out of the candidate points discovered after the pre-processing, using any classical algorithm with minimal overhead.

1.1 RELATED WORKS

Recently, several novel algorithms are developed to obtain CH for a point set. Fu and Lu in [2] propose an improved algorithm which finds convex hull boundary by recursively dividing the set of points into sub regions and updating them after each recursive division. A polynomial-time algorithm for the D-convex hull of a finite point set in the Plane is discussed in [10]. Some algorithms enhance performance though excluding non-convex hull vertexes to reduce the analysis of the minimum convex hull, such as grouping the set of points [9, 11], establishing the auxiliary grid field [12] and obtaining the extreme points as in the quick convex hull [7, 13, 14, 15]. The Floyd quadrilateral method and octagonal methods are some of the quick convex hull algorithms. Huang and Liu [16] present an algorithm

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based on binary tree that builds convex hull with scattered point based on the concept of quick convex hull.

Clarkson, Mulzer and Seshadhri developed a self-improving algorithm to compute planar convex hull points [17]. Liu and Wang [18] propose a reliable and effective CH algorithm based on a technique named Principle Component Analysis for pre-processing the planar point set. A fast CH algorithm with maximum inscribed circle affine transformation is described in [19]. Some algorithms are designed targeted toward harnessing the power of GPUs as in [20].

1.2 PROPOSED ALGORITHM

Our algorithm is based on the fact that in a given set of randomly scattered points, over a two dimensional Euclidean space, only the points to the exterior of the region covered by the point set, participate in constructing the convex hull. Rests of the interior points are not of any importance. So we try to obtain the exterior points near the boundary and check whether they are proper candidates for being convex hull points.

For this purpose we divide the point set into four rectangular region or *p*-Rects (see Fig. 1). The four *p*-Rects are called Bottom-Left (BL), Bottom-Right (BR), Top-Left (TL) and Top-Right (TR). The points inside the *p*-Rects are called points of interest. The four *p*-Rects distribute the entire point set into 5 different clusters based on location of individual points (location based clustering). Each of the clusters contains some edges of the convex hull. Each of the four *p*-Rects is constructed by two corner points which have extreme x/y coordinates chosen as explained next.

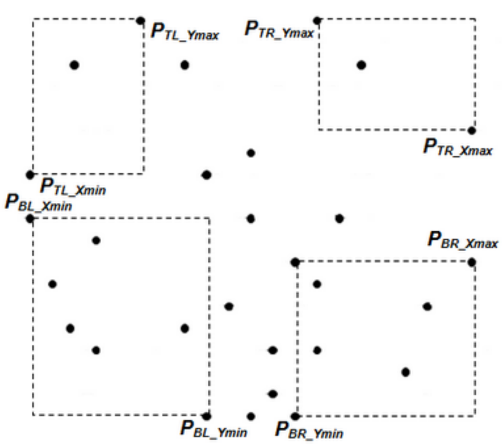


Fig. 1. Four regions (*p*-Rect) with the points of interest constructed by their two corner points

The points in Bottom-Left (BL) region are bounded by a *p*-Rect whose two opposite corner points are P_{BL_Xmin} and P_{BL_Ymin}, where

P_{BL_Xmin} is the point with minimum x-coordinate.

P_{BL_Ymin} is the point with minimum y-coordinate.

The points in Bottom-Right (BR) region are bounded by a *p*-Rect whose two opposite corner points are P_{BR_Xmax} and P_{BR_Ymin}, where

P_{BR_Xmax} is the point with maximum x-coordinate.

P_{BR_Ymin} is the point with minimum y-coordinate.

The points in Top-Left (TL) region are bounded by a *p*-Rect whose two opposite corner points are P_{TL_Xmin} and P_{TL_Ymax}, where

P_{TL_Xmin} is the point with minimum x-coordinate.

P_{TL_Ymax} is the point with maximum y-coordinate.

The points in Top-Right (TR) region are bounded by a *p*-Rect whose two opposite corner points are P_{TR_Xmax} and P_{TR_Ymax}, where

P_{TR_Xmax} is the point with maximum x-coordinate.

P_{TR_Ymax} is the point with maximum y-coordinate.

In case of a conflict between two or more points having same extreme x-coordinate or y-coordinate, we resolve that using the following rules.

If number of points having minimum x-coordinate is greater than 1 then, for Bottom-Left *p*-Rect we take the point with minimum y-coordinate and make it P_{BL_Xmin}. While for Top-Left *p*-Rect we take the point with maximum y-coordinate and make it P_{TL_Xmin}.

If more than one point has maximum x-coordinate then, for Bottom-Right *p*-Rect we take the point with minimum y-coordinate and make it P_{BR_Xmax}. While for Top-Right *p*-Rect the point with maximum y-coordinate is chosen as P_{TR_Xmax}.

If number of points having minimum y-coordinate is greater than 1 then, for Bottom-Left *p*-Rect we take the point with minimum x-coordinate and make it P_{BL_Ymin} and for Bottom-Right *p*-Rect we take the point with maximum x-coordinate and make it P_{BR_Ymin}.

If more than one point has maximum y-coordinate then, for Top-Left *p*-Rect we take the point with minimum x-coordinate and make it P_{TL_Ymax}. While for Top-Right *p*-Rect we take the point with maximum x-coordinate and make it P_{TR_Ymax}.

The above 8 points are directly included in the convex hull candidate point set. There may be less than 8 points in case two extreme points are coincident. For example, P_{BL_Xmin} and P_{TL_Xmin} can be the same point if there is only one point with minimum x-coordinate.

The points which have the minimum/maximum x-coordinates or y-coordinates but do not belong to any of the four *p*-Rects are directly included in the convex hull candidate point set because of the obvious fact that points with extreme x/y coordinate are definitely part of convex hull. These points are identified from the cluster which does not belong to any of the four *p*-Rects. The number of candidate points detected from this cluster is proportional to the size of the cluster and distribution of points over the two dimensional Euclidean space.

A. Processing the Points of Interest

Points inside the p-Rects go through a recursive process which gradually finds out the boundary points in that region. But before explaining the process we present a brief description of a related concept, which determines the clockwise/anti-clockwise orientation of a point with respect to another point.

Let $p_0(x_0, y_0)$, $p_1(x_1, y_1)$ and $p_2(x_2, y_2)$ are three discrete points and we want to determine whether p_2 is clockwise or anticlockwise from p_1 with respect to p_0. For that we calculate cross product between the directed segments p_0p_1 and p_0p_2 like in (1) [1].

$$(p_2 - p_0) \times (p_1 - p_0) = (x_2 - x_0)(y_1 - y_0) - (x_1 - x_0)(y_2 - y_0) \quad (1)$$

If the sign of this cross product is negative, then p_2 is counter-clockwise with respect to p_1. A positive cross product indicates a clockwise orientation (see Fig. 2). A cross product of 0 means that the points p_0, p_1, and p_2 are collinear.

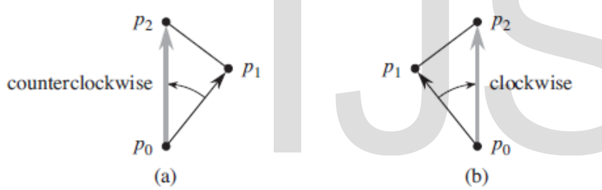


Fig. 2. Using cross product to determine point orientation. (a) Negative cross product means counter clockwise rotation. (b) Positive cross product indicates clockwise rotation.

1) Bottom-Left p-Rect Operation: For this section the points of interest is the set of points $\{p_1, p_2, p_3 ...\}$ where each $p_i = (x_i, y_i)$ such that

$$P_{BL\_Xmin}.x < x_i < P_{BL\_Ymin}.x$$
$$P_{BL\_Ymin}.y < y_i < P_{BL\_Xmin}.y$$

Out of this set we find out P'_{BL_Xmin} and P'_{BL_Ymin} the same way we found P_{BL_Xmin} and P_{BL_Ymin} for the outer Bottom-Left p-Rect. Further processing depends upon the following 3 cases.

Case 1: If there are no such points then we terminate further processing in this p-Rect.

Case 2: If both are same point then we check whether the point is clockwise or anticlockwise from point P_{BL_Ymin} with respect to point P_{BL_Xmin} (see Fig. 3). If it is clockwise or collinear then we include it to the convex hull candidate point set and terminate further processing.

Case 3: If both are different than we check whether they are clockwise or anticlockwise from point P_{BL_Ymin} with respect to point P_{BL_Xmin}. Again we will have 3 scenarios:

- If none of them are clockwise from P_{BL_Ymin} then we leave further processing.
- If only any one of them is clockwise from P_{BL_Ymin} then we include it to the convex hull point set and terminate further processing in this section.
- If both of them are clockwise from P_{BL_Ymin} or collinear then we construct a smaller p-Rect keeping P'_{BL_Xmin} and P'_{BL_Ymin} as corner points and recursively process the points of interest in the new Bottom-Left (BL') p-Rect using above steps in search of new convex hull points.

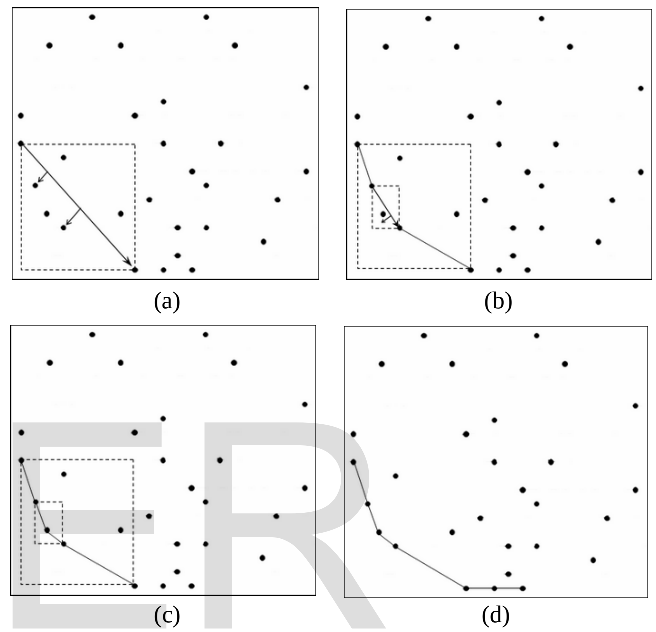


Fig. 3. Operation on the Bottom-Left p-Rect points. (a) Checking orientation of P'_{BL_Xmin} and P'_{BL_Ymin} from P_{BL_Ymin}. (b) Creation of new Bottom-Left (BL') p-Rect and processing points inside. (c) Inclusion of new candidate points inside inner p-Rect. (d) Inclusion of points collinear with P_{BL_Ymin}.

2) Bottom-Right p-Rect Operation: For this section the points of interest is the set of points $\{p_1, p_2, p_3 ...\}$ where each $p_i = (x_i, y_i)$ such that

$$P_{BR\_Ymin}.x < x_i < P_{BR\_Xmax}.x$$
$$P_{BR\_Ymin}.y < y_i < P_{BR\_Xmax}.y$$

From this set we find out P'_{BR_Xmax} and P'_{BR_Ymin} the same way we found P_{BR_Xmax} and P_{BR_Ymin} for outer Bottom-Right p-Rect. Further processing based on these two points is done using similar 3 cases explained in Bottom-Left p-Rect operation.

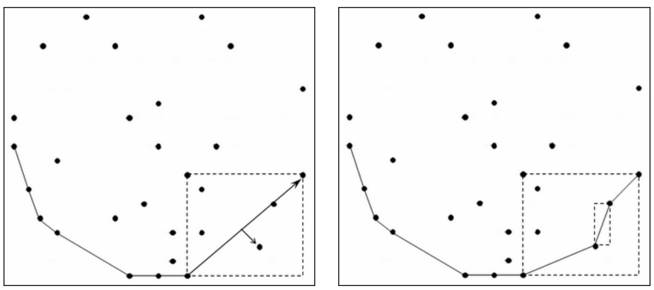
Here the points are checked whether they are clockwise or anticlockwise from point P_{BR_Xmax} with respect to point P_{BR_Ymin} (see Fig. 4).

The recursive operation continues until no new candidate points are found for convex hull.

3) *Top-Right p-Rect Operation:* For this section the points of interest is the set of points $\{p_1, p_2, p_3 \ldots\}$ where each $p_i = (x_i, y_i)$ such that

$$P_{TR\_Ymax}.x < x_i < P_{TR\_Xmax}.x$$
$$P_{TR\_Xmax}.y < y_i < P_{TR\_Ymax}.y$$

Out of this set we find out P'_{TR_Xmax} and P'_{TR_Ymax} the same way we found P_{TR_Xmax} and P_{TR_Ymax} for the outer Top-Right section. Further processing based on these two points is done using 3 cases similar to the cases explained in Bottom-Left *p*-Rect operation.



(a) (b)

Fig. 4. Bottom-Right *p*-Rect operation of the points. (a) Checking orientation of P'_{BR_Xmax} and P'_{BR_Ymin} from P_{BR_Xmax}. (b) Processing of points inside new Bottom-Right (BR') section.

Here the points are checked whether they are clockwise or anticlockwise from point P_{TR_Ymax} with respect to point P_{TR_Xmax} (see Fig. 5).

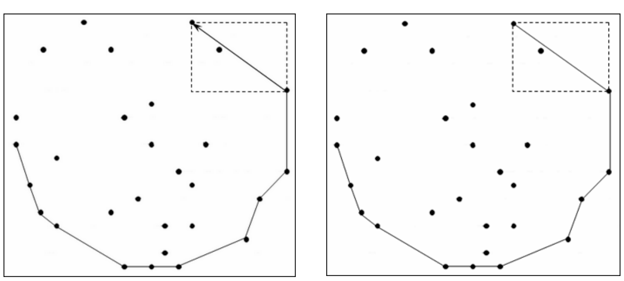


Fig. 5. Top-Right *p*-Rect operation of the points. Here both P'_{TR_Xmax} and P'_{TR_Ymax} are same and it is anticlockwise from P_{TR_Ymax}.

The recursive operation continues until no new points are found eligible for convex hull point set.

4) *Top-Left p-Rect Operation:* For this section the points of interest is the set of points $\{p_1, p_2, p_3 \ldots\}$ where each $p_i = (x_i, y_i)$ such that

$$P_{TL\_Xmin}.x < x_i < P_{TL\_Ymax}.x$$
$$P_{TL\_Xmin}.y < y_i < P_{TL\_Ymax}.y$$

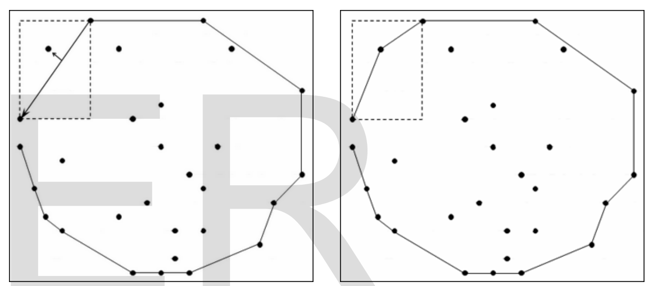
From this set we find out P'_{TL_Xmin} and P'_{TL_Ymax} the same way we found P_{TL_Xmin} and P_{TL_Ymax} for outer Top-Left *p*-Rect. Further processing based on these two points is done using similar 3 cases explained in Bottom-Left section operation.

Here the points are checked whether they are clockwise or anticlockwise from point P_{TL_Xmin} with respect to point P_{TL_Ymax} (see Fig. 6).

The recursive operation continues until no new points are found eligible for convex hull point set.

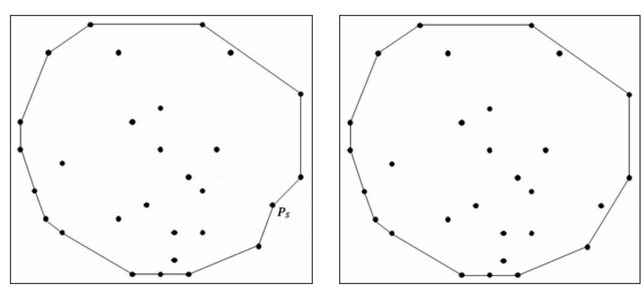
B. Removing the non-convex hull points

After performing the four sectional operations the points we get are called candidate points for convex hull. We call them candidate points because there are some extraneous points which do not actually belong to the convex hull. Figure 7(a) shows such a point P_S. To remove such points we perform standard convex hull algorithms such as Graham's scan or Jarvis's March on the obtained candidate point set (see Figure 7(b)).



(a) (b)

Fig. 6. Operation of the Top-Left *p*-Rect points. (a) Checking orientation of P'_{TL_Xmin} / P'_{TL_Ymax} from P_{TL_Xmin}. (b) Inclusion of the point & termination of search in this section.



(a) (b)

Fig. 7. Removal of non-convex hull point. (a) P_S is a non-convex point. (b) P_S removed after applying Graham's scan.

2 Algorithm Analysis

A. Time Efficiency Analysis

Our algorithm contains four recursive point region operation each of which finds the candidate points for convex hull and hence its running time can be described using a recurrence equation or recurrence. Moreover these four operations are symmetric. So if we find the time complexity of any of the four operations avoiding multiplication to the constant factor 4, that will be time complexity of the entire algorithm.

It is basically a divide-and-conquer algorithm in which the divide operation includes searching the points of interest in a specific region. Combine step includes searching for points with minimum and/or maximum x and/or y coordinates, checking their rotation and keeping the distinct candidate points in a storage (e.g. stack) for further processing by standard convex hull algorithms. In conquer step we recursively solve a sub problem of 1/b the size of original input point set. The value of b can be anything depending upon orientation of the points, without affecting the final time complexity.

For the two search operations if we apply linear search it requires $O(n)$ time on n points. $O(1)$ time is required for both rotation checking and storing the candidate points. So the entire combine step takes $O(n)$ time as shown in (2).

$$C(n) = O(n) + 2.O(1) = O(n) \quad (2)$$

Running time for Divide step is

$$D(n) = O(n) \quad (3)$$

So the recurrence for the worst-case running time $T(n)$ of the algorithm is:

$$T(n)=\begin{cases} O(1) & \text{if } n \le 3 \\ T\left(\frac{n}{b}\right) + O(n) & \text{if } n > 3 \end{cases} \quad (4)$$

Since

$$C(n) + D(n) = O(n) + O(n) = O(n) \quad (5)$$

Solution to this recurrence is $T(n) = O(n \lg n)$. As we can see the prime factor affecting the time complexity of the preprocessing algorithm is the time required by the point searching operation. As we are applying a linear search in our algorithm, its time complexity is not dependent on the randomness and the entropy of the 2-D point set. If we apply a more efficient searching algorithm such as binary tree search running time can be reduced to $O(\lg n \lg n)$.

Let after the preprocessing step we get m candidate points, where $m << n$ and n is very large. So for processing by standard convex hull algorithms like Graham's scan ($O(n \lg n)$) and Jarvis's march ($O(nh)$, h is the number of points in convex hull), the required time will be quite less, $O(m \lg m)$ and $O(mh)$ respectively.

Hence if $T'(n)$ is the running time of our preprocessing algorithm and $T(m)$ is the running time of any convex hull algorithm, then the total running time $T(n) = T'(n) + T(m)$ will be less than that of the convex hull algorithm applied on non preprocessed point set.

B. Space Efficiency Analysis

This divide-and-conquer is not much space efficient, as in each recursive call we rest and store them separately for further processing. As a result in each recursive call a new memory space of size $(1/b)^{th}$ of the memory space of previous step is reserved.

C. Test of Numeral Value

We developed a simulation of our proposed algorithm using C programming language and tested it over several configurations of scattered points. We are presenting the test results in the following tables. The Graham's scan algorithm implementation uses merge sort, while Jarvis's march uses linear search for finding the next pivot point.

In Tab.1. shows a particular case of processing 10,000 points for Bottom-Left section. As we can see after each level of recursion, number of points of interest reduces quickly with great speed of convergence. This condition also holds for other larger amount of point set. In fact in our simulations we found very few cases when the process reached 4^{th} level of recursion.

	points of interest	**Candidate points found**	**Total candidate points**
original data	10000	2	2
level-1 recursion	603	2	4
level-2 recursion	7	2	6
level-3 recursion	2	1	7

Tab. 1. Statistics of Point Reduction after Each Level of Recursion.

In Tab. 2. gives a comparative running time (in ms) analysis by several algorithms under our test environment. It is very clear that with the growing number of input points the efficiency of our algorithm gets better. The preprocessing algorithm narrows down the amount of candidate points to such an extent that the applied Graham's scan or Jarvis's march completes very quickly.

Data amount	**Jarvis march**	**Graham scan**	**Proposed preprocessing + Jarvis march**	**Proposed preprocessing + Grahams**
10000	16	83	12	34

30000	56	1243	15	81
50000	94	4344	34	169
100000	187	7323	53	410

Tab.2. Performance Comparision with other Algorithms(MS).

In Tab. 3. we present a comparative study of average performance gain over the classical Graham scan and Jarvis march algorithms by some of the recent convex hull algorithms presented in [16][21][22]. All these algorithms use almost similar principle for convex hull computation and the performance gain is computed based on the computation time required by them. The study shows that in most of the cases our proposed algorithm gives the highest performance gain.

Algorithm	Graham Scan	Jarvis March
e-Quad based convex Hull [21]	2.36%	94.05%
Proved Quick Convex Hull [16]	82.73%	48.16%
Fast Convex Hull [22]	79.58%	
Our Proposed algorithm	95.67%	58.4%

Tab. 3. Comparision of Performance Gain.

3 CONCLUSION

In this paper we have discussed about a preprocessing algorithm for finding convex hull points, which is based on the fact that only the points existing to the boundary of a region covered by a randomly scattered set of points, take part in constituting the convex-hull. We have discussed about its running time complexity and also proved through real-time simulation results that this preprocessing improves the overall running time of convex hull algorithms. Although the purpose of developing this algorithm was to find all valid convex hull points and use it as a standard Quick Convex Hull algorithm. But as we saw that the algorithm outputs some invalid convex hull points too and removing them was not possible, keeping the running time constraint into consideration. So, we decided to keep it as a preprocessing step. Our future work will be focusing on how we can remove this deficiency and also reduce the space complexity so that it can be used as a standalone convex hull algorithm.

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__Ngangbam Herojit Singh__ was born in Imphal West, Manipur,India. He received his M.Tech. from Anna University Chennai in 2013.Currently,he is working as Research Assistant in National Institute of Technology,Manipur. His research interests NetworksSecurity,Algorithms, Pattern Recognitio .(Email ID:herojitng@gmail.com).

__Laiphrakpam Dolendro Singh__ was born in Imphal East, Manipur,India. He received his B.E. from Anna University Chennai in 2011.Currently,he is pursuing M.E from National Institute of Technology,Agartala. His research interests include Networking,Image processing.(Email ID:doltana@gmail.com).